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NOTE

On the Accuracy and Stability of Explicit Schemes for Multidimensional Linear Homogeneous Advection Equations

1. INTRODUCTION

In [1] Roe provided, in a simple form, conditions which determine the accuracy of a numerical scheme for the solution of the one-dimensional linear advection equation

$$u_t + au_x = 0, \tag{1}$$

where a is a constant wavespeed, by considering a general scheme for (1) in the form

$$u_i^{n+1} = \sum_\alpha A_\alpha u_{i+\alpha}^n, \tag{2}$$

where $u_i^n = u(i\,\Delta x, n\,\Delta t)$, $\{A_\alpha\}$ is a finite set of constant, nonzero coefficients, Δx is the constant mesh spacing, and Δt is the timestep. Define the Courant number as $\nu = a\,\Delta t/\Delta x$. Roe proved the following two theorems.

THEOREM 1. *If u_i^n is a polynomial of degree p in i, scheme* (2) *will give the exact solution to* (1) *if and only if*

$$\sum_\alpha \alpha^q A_\alpha = (-\nu)^q \tag{3}$$

for all integers q such that $0 \le q \le p$.

THEOREM 2. *If scheme* (2) *meets the conditions of Theorem* 1, *the leading term in its pointwise error is*

$$\frac{a\,\Delta x^p}{\nu(p+1)!}\left[(-\nu)^{p+1} - \sum_\alpha \alpha^{p+1}A_\alpha\right]\frac{\partial^{p+1}}{\partial x^{p+1}}u. \tag{4}$$

These two theorems enabled the following definition.

DEFINITION 1. Any scheme of the form (2) for the one-dimensional linear advection equation (1) *that satisfies the conditions in Theorem* 1 *is called* p^{th}-order accurate in space and time.

The aim of this paper is to extend Roe's result to two and three dimensions. Methods of finding stability restrictions on multidimensional schemes are also discussed.

2. THE TWO-DIMENSIONAL CASE

In two space dimensions the linear advection equation can be written as

$$u_t + au_x + bu_y = 0, \tag{5}$$

where a and b are constant wavespeeds. Schemes for (5) on regular, Cartesian grids in the form

$$u_{i,j}^{n+1} = \sum_{\alpha,\beta} A_{\alpha,\beta} u_{i+\alpha,j+\beta}^n \tag{6}$$

will be considered, where $u_{i,j}^n = u(i\,\Delta x, j\,\Delta y, n\,\Delta t)$, Δx and Δy define the mesh spacing in the x and y directions, and $\{A_{\alpha,\beta}\}$ form a finite set of constant, nonzero coefficients. Define the directional Courant numbers $\nu_x = a\,\Delta t/\Delta x$ and $\nu_y = b\,\Delta t/\Delta y$. In two dimensions, Theorem 1 becomes

THEOREM 3. *If $u_{i,j}^n$ is a polynomial of degree p in i and j, scheme* (6) *will give the exact solution of* (5) *if and only if*

$$\sum_{\alpha,\beta} \alpha^q \beta^r A_{\alpha,\beta} = (-\nu_x)^q(-\nu_y)^r \tag{7}$$

for all integer pairs (q, r) such that $q \ge 0$, $r \ge 0$ and $q + r \le p$.

Proof. Choose a basis for the set of two-dimensional polynomials as

$$\{x^q y^r\}_{q,r} \tag{8}$$

for integers q, r. It is only necessary to consider the performance of the scheme on the general basis element; i.e., set

$$u_{\alpha,\beta}^n = (\alpha\,\Delta x)^q(\beta\,\Delta y)^r. \tag{9}$$

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Without loss of generality, only the origin (0, 0) is considered. Performing one timestep of (6) at (0, 0) gives

$$u_{0,0}^{n+1} = \sum_{\alpha,\beta} A_{\alpha,\beta}(\alpha\,\Delta x)^q(\beta\,\Delta y)^r. \tag{10}$$

The exact solution is

$$u_{0,0}^{n+1} = u(-a\,\Delta t, -b\,\Delta t, n\,\Delta t) = (-a\,\Delta t)^q(-b\,\Delta t)^r. \tag{11}$$

Therefore the scheme gives the exact solution if and only if

$$\sum_{\alpha,\beta} A_{\alpha,\beta}(\alpha\,\Delta x)^q(\beta\,\Delta y)^r = (-a\,\Delta t)^q(-b\,\Delta t)^r; \tag{12}$$

i.e.,

$$\sum_{\alpha,\beta} \alpha^q\beta^r A_{\alpha,\beta} = (-\nu_x)^q(-\nu_y)^r \tag{13}$$

for all (q, r) such that $q \geq 0$, $r \geq 0$ and $q + r \leq p$. This is the required result, and the theorem is proved.

THEOREM 4. *If scheme* (6) *meets the conditions in Theorem* 3, *then the leading terms in the pointwise error have the form*

$$\frac{1}{\Delta t}\frac{\Delta x^q}{q!}\frac{\Delta y^{p+1-q}}{(p+1-q)!}\cdot D_q \cdot u^n_{(qx)((p+1-q)y)}, \tag{14}$$

where

$$D_q = (-\nu_x)^q(-\nu_y)^{p+1-q} - \sum_{\alpha,\beta}\alpha^q\beta^{p+1-q}A_{q,p+1-q} \tag{15}$$

for some integer q such that $0 \leq q \leq p + 1$, *and*

$$u^n_{(qx)(ry)} = \frac{\partial}{\partial x^q \partial y^r} u^n. \tag{16}$$

Proof. The pointwise error is defined as the difference between the exact solution and the numerical solution divided by the timestep. Without loss of generality, consider the origin (0, 0). The pointwise error here is

$$P(0,0) = \frac{1}{\Delta t}[u(0,0,(n+1)\,\Delta t) - u_{0,0}^{n+1}] \tag{17}$$

$$= \frac{1}{\Delta t}\left[u(-a\,\Delta t, -b\,\Delta t, n\,\Delta t) - \sum_{\alpha,\beta} A_{\alpha,\beta}u^n_{\alpha,\beta}\right]. \tag{18}$$

Expanding both terms in brackets as Taylor series gives

$$P(0,0) = \frac{1}{\Delta t}\left[\sum_{q,r}\frac{(-a\,\Delta t)^q}{q!}\frac{(-b\,\Delta t)^r}{r!}u^n_{(qx)(ry)} - \sum_{\alpha,\beta}A_{\alpha,\beta}\left(\sum_{q,r}\frac{(\alpha\,\Delta x)^q}{q!}\frac{(\beta\,\Delta y)^r}{r!}u^n_{(qx)(ry)}\right)\right], \tag{19}$$

where the derivatives $u^n_{(qx)(ry)}$ are taken at the origin. Rearranging gives

$$P(0,0) = \frac{1}{\Delta t}\left[\sum_{q,r}\frac{(-a\,\Delta t)^q}{q!}\frac{(-b\,\Delta t)^r}{r!} \times \left(1 - \frac{1}{\nu_x^q}\frac{1}{\nu_y^r}\sum_{\alpha,\beta}(-\alpha)^q(-\beta)^r A_{\alpha,\beta}\right)u^n_{(qx)(ry)}\right]. \tag{20}$$

If the conditions in Theorem 3 hold, then the terms in (20) with $q + r \leq p$ vanish, and the first terms in the pointwise error have the form

$$\frac{1}{\Delta t}\frac{(-a\,\Delta t)^q}{q!}\frac{(-b\,\Delta t)^r}{(r)!} \times \left(1 - \frac{1}{\nu_x^q}\frac{1}{\nu_y^r}\sum_{\alpha,\beta}(-\alpha)^q(-\beta)^r A_{\alpha,\beta}\right)u_{(qx)(ry)}, \tag{21}$$

where $q + r = p + 1$. This can be written as (14), (15) by using the identity $r = p + 1 - q$ and rearranging. Hence the theorem is proved.

Theorems 3 and 4 enable the following definition.

DEFINITION 2. Any scheme of the form (6) for the two-dimensional linear advection equation (5) that satisfies the conditions in Theorem 3 is called p^{th}-order accurate in space and time.

COROLLARY 1. *Any scheme of the form* (6) *for the two-dimensional linear advection equation* (5) *must satisfy*

$$N = \sum_{i=1}^{p+1} i \tag{22}$$

conditions for p^{th}*-order accuracy, and therefore must have a stencil of at least N points.*

Proof. Follows easily from Theorem 3.

3. THE THREE-DIMENSIONAL CASE

In three space dimensions the linear advection equation can be written as

$$u_t + au_x + bu_y + cu_z = 0, \tag{23}$$

where a, b, and c are constant wavespeeds. Schemes for (5) of the form

$$u^{n+1}_{i,j,k} = \sum_{\alpha,\beta,\gamma} A_{\alpha,\beta,\gamma} u^n_{i+\alpha,j+\beta,k+\gamma} \tag{24}$$

will be considered, where $u^n_{i,j,k} = u(i\,\Delta x, j\,\Delta y, k\,\Delta z, n\,\Delta t)$, Δx, Δy, and Δz define the mesh spacing in the x, y, and z directions, respectively, and $\{A_{\alpha,\beta,\gamma}\}$ form a finite set of constant, nonzero coefficients. Define directional Courant numbers as $\nu_x = a\,\Delta t/\Delta x$, $\nu_y = b\,\Delta t/\Delta y$, and $\nu_z = c\,\Delta t/\Delta z$. In this case Theorem 1 becomes

THEOREM 5. *If $u^n_{i,j,k}$ is a polynomial of degree p in i, j, and k, scheme* (24) *will give the exact solution of* (23) *if and only if*

$$\sum_{\alpha,\beta,\gamma} \alpha^q \beta^r \gamma^s A_{\alpha,\beta,\gamma} = (-\nu_x)^q(-\nu_y)^r(-\nu_z)^s \tag{25}$$

for all integer triples (q, r, s) such that $q \geq 0$, $r \geq 0$, $s \geq 0$, and $q + r + s \leq p$.

THEOREM 6. *If scheme* (24) *meets the conditions in Theorem* 5, *then the leading terms in the pointwise error have the form*

$$\frac{1}{\Delta t}\frac{\Delta x^q}{q!}\frac{\Delta y^r}{r!}\frac{\Delta z^{p+1-q-r}}{(p+1-q-r)!}\cdot D_{q,r}\cdot u^n_{(qx)(ry)((p+1-q-r)z)}, \tag{26}$$

where

$$\begin{aligned} D_{q,r} = {} & (-\nu_x)^q(-\nu_y)^r(-\nu_z)^{p+1-q-r} \\ & - \sum_{\alpha,\beta,\gamma} \alpha^q\beta^r\gamma^{p+1-q-r} A_{q,r,p+1-q-r} \end{aligned} \tag{27}$$

for some integer pair (q, r) such that $q \geq 0$, $r \geq 0$, $0 \leq q + r \leq p + 1$, and

$$u^n_{(qx)(ry)(sz)} = \frac{\partial}{\partial x^q\,\partial y^r\,\partial z^s}u^n. \tag{28}$$

The proof of Theorems 5 and 6 follows easily from the two-dimensional case. They lead to the definition.

DEFINITION 3. Any scheme of the form (24) for the three-dimensional linear advection equation (23) that satisfies the conditions in Theorem 5 is called p^{th}-order accurate in space and time.

COROLLARY 2. *Any scheme of the form* (24) *for the three-dimensional linear advection equation* (23) *must satisfy*

$$N = \sum_{j=1}^{p+1}\left(\sum_{i=1}^{j} i\right) \tag{29}$$

conditions for p^{th}-order accuracy, and therefore must have a stencil of at least N points.

Proof. Follows easily from Theorem 5.

4. STABILITY

The stability of advection schemes is obviously as important as their accuracy. However, the application of standard stability analysis techniques, such as von Neumann stability analysis, to schemes in two and three dimensions is notoriously hard, due to the complexity of algebraic expressions encountered when applying these techniques in multiple dimensions. Indeed, even determining the stability of a scheme in one space dimension can be hard if the scheme has a large stencil, or has many unspecified parameters in its coefficients. In the previous sections we have derived conditions on the coefficients of general explicit schemes that determine the accuracy of multidimensional schemes for linear advection; one would hope it is possible to derive fairly simple conditions on the coefficients of general schemes that determine stability restrictions on those schemes. To our knowledge, such general conditions do not yet exist in the literature. The authors have derived general conditions for three-point explicit centered schemes for advection–diffusion in one space dimension [2]; however an extension of these conditions to general n-point schemes or multiple dimensions has not yet been found. The simplest approach known to the authors for obtaining a reliable indication of the stability of a scheme when the algebra associated with standard techniques becomes intractable has been presented in [3, 4]. We now briefly describe this technique. Consider the case of a two-dimensional scheme in which the coefficients depend only on the parameters ν_x and ν_y. An algebraic expression for the von Neumann amplification factor S for such schemes can be found, though some manipulation may be required. For a given pair (ν_x, ν_y) one can numerically evaluate S for many thousands of pairs θ, ϕ of phase angles in the x and y directions, and record the *proportion* $p(\nu_x, \nu_y)$ of these pairs for which $S \leq 1$, i.e., for which the scheme is stable. If $p(\nu_x, \nu_y) = 1$, we assume the scheme is stable; otherwise it is unstable. This can be repeated for many pairs (ν_x, ν_y) in a region slightly larger than the expected "stability region(s)" of the scheme. A contour plot of $p(\nu_x, \nu_y)$ in the $\nu_x - \nu_y$ plane then indicates the stability region(s) of the scheme: these will be the region(s) where $p(\nu_x, \nu_y)$ is constant and equal to 1. The extension of this approach to schemes whose coefficients depend on different parameters, and/or to schemes in three dimensions, is obvious. See [3, 4] for application to specific schemes.

5. EXAMPLES

The author has found the accuracy conditions discussed in sections 2 and 3 useful for determining the accuracy of schemes for linear advection. For example, consider the two-dimensional scheme of LeVeque [5] applied to the linear advection equation (5) *with positive speeds*:

$$\begin{aligned} u_{i,j}^{n+1} = (1 - \nu_x^2 - \nu_y^2 + \nu_x\nu_y)u_{i,j}^n \\ - \tfrac{1}{2}\nu_x(1-\nu_x)u_{i+1,j}^n - \tfrac{1}{2}\nu_y(1-\nu_y)u_{i,j+1}^n \\ + \tfrac{1}{2}\nu_x(1+\nu_x-\nu_y)u_{i-1,j}^n + \tfrac{1}{2}\nu_y(1+\nu_y-\nu_x)u_{i,j-1}^n \\ + \nu_x\nu_y u_{i-1,j-1}^n. \end{aligned} \tag{30}$$

Application of the accuracy conditions (7) to this scheme shows that it is second-order accurate in space and time. Using the technique discussed in the last section, this scheme is found to be stable if $\nu_x \leq 1$ and $\nu_y \leq 1$. Consider the three-dimensional scheme [3, 4] applied to (23) *with positive speeds*:

$$\begin{aligned} u_{i,j,k}^n = u_{i,j,k}^n - \tfrac{1}{4}[\nu_x^2(2-\nu_y)(2-\nu_z) + (2-\nu_x)\nu_y^2(2-\nu_z) \\ + (2-\nu_x)(2-\nu_y)\nu_z^2]u_{i,j,k}^n \\ + \tfrac{1}{8}\nu_x[(1+\nu_x)(2-\nu_y)(2-\nu_z) - 2\nu_y^2(2-\nu_z) - 2(2-\nu_y)\nu_z^2]u_{i-1,j,k}^n \\ + \tfrac{1}{8}\nu_y[(2-\nu_x)(1+\nu_y)(2-\nu_z) - 2\nu_x^2(2-\nu_z) - 2(2-\nu_x)\nu_z^2]u_{i,j-1,k}^n \\ + \tfrac{1}{8}\nu_z[(2-\nu_x)(2-\nu_y)(1+\nu_z) - 2(2-\nu_x)\nu_y^2 - 2\nu_x^2(2-\nu_y)]u_{i,j,k-1}^n \\ + \tfrac{1}{8}\nu_x\nu_y[(2-\nu_z)(2+\nu_x+\nu_y) - 2\nu_z^2]u_{i-1,j-1,k}^n \\ + \tfrac{1}{8}\nu_x\nu_z[(2-\nu_y)(2+\nu_x+\nu_z) - 2\nu_y^2]u_{i-1,j,k-1}^n \\ + \tfrac{1}{8}\nu_y\nu_z[(2-\nu_x)(2+\nu_y+\nu_z) - 2\nu_x^2]u_{i,j-1,k-1}^n \\ + \tfrac{1}{8}\nu_x\nu_y\nu_z(3+\nu_x+\nu_y+\nu_z)u_{i-1,j-1,k-1}^n \\ - \tfrac{1}{8}\nu_x(1-\nu_x)(2-\nu_y)(2-\nu_z)u_{i+1,j,k}^n - \tfrac{1}{8}\nu_x(1-\nu_x)\nu_y(2-\nu_z)u_{i+1,j-1,k}^n \\ - \tfrac{1}{8}\nu_x(1-\nu_x)(2-\nu_y)\nu_z u_{i+1,j,k-1}^n - \tfrac{1}{8}\nu_x(1-\nu_x)\nu_y\nu_z u_{i+1,j-1,k-1}^n \\ - \tfrac{1}{8}(2-\nu_x)\nu_y(1-\nu_y)(2-\nu_z)u_{i,j+1,k}^n - \tfrac{1}{8}\nu_x\nu_y(1-\nu_y)(2-\nu_z)u_{i-1,j+1,k}^n \\ - \tfrac{1}{8}(2-\nu_x)\nu_y(1-\nu_y)\nu_z u_{i,j+1,k-1}^n - \tfrac{1}{8}\nu_x\nu_y(1-\nu_y)\nu_z u_{i-1,j+1,k-1}^n \\ - \tfrac{1}{8}(2-\nu_x)(2-\nu_y)\nu_z(1-\nu_z)u_{i,j,k+1}^n - \tfrac{1}{8}(2-\nu_x)\nu_y\nu_z(1-\nu_z)u_{i,j-1,k+1}^n \\ - \tfrac{1}{8}\nu_x(2-\nu_y)\nu_z(1-\nu_z)u_{i-1,j,k+1}^n - \tfrac{1}{8}\nu_x\nu_y\nu_z(1-\nu_z)u_{i-1,j-1,k+1}^n. \end{aligned} \tag{31}$$

Application of conditions (25) to this scheme shows that it is also second-order accurate in space and time. It is left to the reader to verify that the application of the accuracy conditions given here is far less time consuming than applying truncation error analysis directly to the schemes. Applying the technique discussed in the last section indicates that the scheme is stable provided $\max\{\nu_x, \nu_y, \nu_z\} \leq \frac{2}{3}$. See [3] or [4] for the relevant contour plots.

6. SUMMARY

Conditions on the coefficients of schemes for the solution of the two and three dimensional linear advection equations that guarantee the schemes are p^{th}-order accurate in space and time have been presented. Two examples of schemes have been given, one in two dimensions and one in three dimensions, where these accuracy conditions have proven useful. A technique that gives a good indication of the stability conditions of a scheme when conventional methods prove intractable has also been described.

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